Separation Science and Technology
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713708471

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To cite this Article Dixon, D. C.(1978) 'Momentum-Balance Aspects of Free-Settling Theory. III. Transient Compression Resistance', Separation Science and Technology, 13: 9, 753 — 766
To link to this Article: DOI: 10.1080/01496397808057126
URL: http://dx.doi.org/10.1080/01496397808057126

SEPARATION SCIENCE AND TECHNOLOGY, 13(9), pp. 753–766, 1978

Momentum-Balance Aspects of Free-Settling Theory. III. Transient Compression Resistance

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Abstract

The previous analysis of free settling is extended by considering situations involving concentration gradients in the free-settling zone. It is shown that anomalies which arise in the theory can be removed if the normal concept of a free-settling suspension is modified. This involves recognizing that, even when the flocs are not in contact, a suspension will exhibit a transient resistance to concentration change.

INTRODUCTION

A previous discussion of free-settling theory (*1, 2*) has led to conclusions which are contrary to some previously accepted ones. The present discussion further develops the argument so as to include more general situations.

Since there are some differences in the precise meaning of the term "free settling" among different writers, an explanation of its use here is required. For a given suspension (or "sludge" or "slurry"), particle concentration divides into two ranges, "free settling" and "compression," and the "critical concentration" is the boundary between the two. At the critical concentration the flocs (or "particles" in an unflocculated suspension) just touch each other. Below this concentration (i.e., in the free-settling range) the flocs do not exert any forces on each other, and so cannot transmit a compressive stress. Above the critical concentration (i.e., in the compression range) the flocs do exert forces on each other and, to increase the concentration above the critical, compressive stress must be applied to the particle phase. In the free-settling range the particles are subject to two forces only; the gravitational force (allowing for buoyancy) and the drag force due to relative motion between particles and fluid. In the compression range a compressive stress acts between the particles, in addition to the gravitational and drag forces, and its gradient is an additional factor which affects the motion of the particles.

In discussing batch settling of a suspension, starting with a uniform concentration in the free-settling range (*1*), the conclusion reached previously is that no concentration gradient develops in the free-settling zone. Rather, a concentration gradient can only develop in the compression zone (the sediment), and throughout the process there is a concentration and velocity discontinuity between the sediment zone and the free-settling zone, the latter remaining at the initial concentration. As particles pass through this interface, they are retarded by impact with the top of the sediment, and they jump to the lowest compression-range concentration (the critical concentration), which is the concentration at the top of the sediment.

The same arguments applied to steady-state continuous thickening (*2*) show that, contrary to previously accepted concepts, there is no thickening-zone capacity limitation associated with the free-settling concentration range, the particles thickening to the critical concentration as they strike the top of the sediment. This process is not affected by the relative positions of the "operating" and "flux" lines in the free-settling range. Further thickening occurs as the particles pass downward through the sediment toward the sludge outlet due to the increasing compressive stress exerted by the particles above. Thickening as far as the critical concentration requires no compressive stress because the suspension exhibits no resistance to compression in the free-settling range, and this statement led to the comment (*2*) that the concept of free settling (as defined above) is itself an approximation.

When more complicated free-settling situations are considered, involving nonuniform initial concentration, it is found that anomalies occur in the theory, and these are discussed below. The purpose of this paper is to show that it is necessary to acknowledge that a slurry will exhibit a resistance (however small and transient) to change in concentration, no matter how small the concentration. Recognition of this fact removes the anomalies alluded to above and increases the overall unity of thickening theory.

ANOMALIES IN FREE-SETTLING THEORY

Free Settling with an Initially Nonuniform Concentration

The writer has previously argued (*1*) that sedimentation of an initially uniform suspension, starting in free settling, will always be "Type I"; that is, no concentration gradient develops in the free-settling zone. This is contrary to the long-accepted Kynch analysis (*3*), and the immediate question which arises is: "What happens when the suspension is initially in free settling, but with nonuniform concentration?" That is, if one considers the case where a concentration gradient is present initially, how do various concentration planes move through the suspension, and how do these movements compare with those predicted by Kynch's theory?

However, when one attempts to answer this question, an unexpected (by the writer, at least) difficulty arises. After setting up the equations for the nonuniform free-settling zone, one finds that there are insufficient boundary conditions available. Thus the equations cannot be solved and the question posed above cannot be answered.

As in previous discussions, an unnecessary complication can be omitted by assuming that the sediment formed during the process is incompressible, and so is uniform at the critical concentration, and all the particles in it are stationary. The material- and momentum-balance equations for the free-settling zone ($\tau = 0$) are (*1*):

$$\left(\frac{\partial f}{\partial t}\right)_s = -f^2\left(\frac{\partial v}{\partial s}\right)_t \tag{1}$$

and

$$\rho\left(\frac{\partial v}{\partial t}\right)_s = F_g + F_d \tag{2}$$

These are Lagrangian equations, based on particle motions. If the top surface of the free-settling zone is taken as datum ($s = 0$), then the lower boundary of the zone is moving (s = variable). At the lower boundary, particles pass from the free-settling zone into the sediment zone, as they are retarded by impact with the sediment. In general, there is a concentration (and velocity) discontinuity at the free-settling/sediment interface.

Since Eqs. (1) and (2) are spatially first order in v, a v boundary condition is required. However, no such boundary condition is available. The velocity is not known anywhere in the free-settling zone. When an initially uniform suspension is considered (as in previous discussions), this difficulty does not arise. In that case $(\partial v/\partial s)_t$ is zero, Eq. (1) does not have to be used, and v can be obtained as a function of time from Eq. (2).

The reason for the lack of the necessary boundary condition in the present problem is the discontinuity at the free-settling/sediment interface. On the underside of this discontinuity, velocity and concentration are known, but on the upper side neither is known because the free-settling suspension can jump from any concentration and velocity to the critical concentration. Neither velocity nor concentration is known at either end of the free-settling zone. The ability to jump in concentration is due to the lack of compression resistance in the free-settling range.

Fluidization of a Free-Settling Suspension

It seems that the boundary-condition difficulty found in the preceding example can be avoided by considering the behavior of a nonuniform, free-settling suspension in the presence of a steady, uniform liquid upflow, introduced through a porous container bottom. The upflow rate can be chosen so that at steady state the concentration is uniform at a value which is less than the critical concentration. That is, the particles are supported entirely by liquid drag, and no compressive stress exists. Then, by considering the suspension to start with nonuniform concentration a little displaced from the steady-state concentration, free settling of a nonuniform suspension is obtained without the formation of a sediment.

In this situation the necessary boundary condition on v is available. At the bottom of the column, on the porous support plate, the particle velocity is zero. Thus, using the same description of the suspension and two different, but both physically realistic, circumstances, the equations and boundary conditions are available for one case but not for the other. This is a contradictory result.

However, the present example, while not suffering from lack of a boundary condition, presents a different difficulty. The solution of the equations indicates that the upflow system is always unstable, but this does not accord with experimental evidence. While some fluidized systems exhibit behavior which could be the result of instability ("aggregative" fluidization), not all systems exhibit such behavior, and, in particular, liquid-fluidized systems usually appear to be quite stable (*4*). [While Eqs. (1) and (2) cannot be solved analytically, in general, linearized analytical solutions can be obtained, and are sufficient to indicate the stability of the system. The instability of the linearized solutions is shown in the Appendix. The analytical procedure used is similar to, and reaches the same conclusion as, those of Jackson (*4*) and Pigford and Baron (*5*), but it makes use of the available boundary condition for the system.]

PERMANENT AND TRANSIENT COMPRESSIVE STRESSES

The anomalies in free-settling theory, illustrated by the two examples above, are removed when a transient resistance to compression is included in the analysis. This is shown below, after discussing the factors which affect the interparticle compressive stress in a suspension.

The usual concept is that the critical concentration marks the start of compressive stress action, and, therefore, of resistance to compression. The critical concentration is envisaged as the concentration at which the flocs come into mechanical contact. To produce concentrations greater than the critical, compressive stress must be applied, and the simplest assumption to make is that the stress required depends only on the concentration (for a given slurry). Thus, below the critical concentration the stress is zero while above the critical concentration the stress is a function of concentration. (In terms of cause and effect, the concentration is a function of stress, rather than the other way around, but this is no different analytically.)

However, it is clear that the concentration achieved in compressing a sludge does not depend only on the applied stress. It will also depend in a complex way on numerous factors which determine the geometrical arrangement of the particles. For this reason, care is taken to minimize disturbance of the samples in compression testing of soils (*6*), since this affects the results. The possible large effect of particle arrangement is easily demonstrated by considering the stacking of spherical particles.

Equal-sized spheres can be stacked with porosity of 0.476 in cubical packing and 0.260 in rhombohedral packing, and can also be stacked stably at a porosity at least as high as 0.875 (*7*). If two columns containing equal weights (say 10 kg) of the same hard, equal-sized spheres are considered, one stacked in cubical packing and one in rhombohedral, the compressive stress acting at the bottom of each column will be the same (neglecting wall effects). However, the particle concentrations are very different, and clearly the concentration is virtually independent of the stress (being affected very slightly by the distortion of the spheres under the weight above), depending almost entirely on the packing arrangement.

However, in the case of flocculated suspensions the individual entities (the flocs) are much more compressible than in the case of separate hard particles, and the effect of variations in packing arrangement will be greatly reduced if the packing process is "randomized" in some sense. The compression of flocs does not involve distortion of the individual particles to any great extent, but rather overcoming the interparticle forces in moving the particles closer together. Thus, at least until the particles are approaching the close-packed condition, it is to be expected that the concentration achieved will depend primarily on the stress applied, provided that variations in packing can be minimized. This is borne out by experimental sedimentation data (*8–10*).

It is to be expected that slow stirring of a suspension is necessary to prevent bridge formation and so reduce packing variation effects, and this is shown by the results of Dell and Keleghan (*9*). Shannon et al. (*11*), settling glass spheres, only obtained a reproducible final concentration when the cylinder was vibrated by rapping the side. On the other hand, Shin and Dick (*10*) found concentration to be a function of stress even in an unstirred suspension. Nevertheless, it seems clear that, to obtain consistent results, it is necessary to avoid as far as possible effects due to variations in packing arrangement, and that a large part of the scatter in experimental results is due to such effects.

Thus, while recognizing its limitations, the assumption that concentration achieved depends only on stress applied is usually adopted. However, there is a further factor involved in the compression process, namely a transient compression resistance, depending on the rate of change of concentration. Since no process can occur instantaneously, application of stress to a portion of sludge will not instantaneously produce the corresponding concentration. There will be a time delay, or dynamic effect, involved in the process.

While it is not essential to the present argument to determine the source of the transient compression resistance, at least one effect of this type is easily identified. As particles are moved closer together (which is what happens during compression), liquid is "squeezed out" from between them, and the flow resistance of the liquid gives rise to a force resisting the compression. This is easily seen in the simple case of two spheres moving vertically toward each other with equal velocities in a stationary liquid.

Clearly, each sphere experiences an equal force, upward on the upper, and downward on the lower. That is, a repulsive force acts between the spheres. [An analytical expression for this force can be obtained for creeping flow (*12*). The similar situation involving two disks can also be analyzed (*13*).] This force is distinct from the usual drag force due to relative motion of particles and liquid. In the present case the velocity of the two spheres, taken together, is zero, the net force is zero, and the drag is zero. If the two spheres were moving with unequal velocities, each would experience a drag force (the same magnitude and sign for each) depending on their average velocity, and a repulsive force (the same magnitude but opposite signs) depending on their velocity of approach to each other.

In a suspension or sludge the situation is obviously much more complex, but nevertheless it is clear that qualitatively the same effect will occur. The slurry will exhibit a transient resistance to compression which depends on the rate of increase in concentration (and on the concentration itself and other slurry properties). Thus, at each point in the suspension, the compressive stress can be split into two parts, the "permanent" or "static" component, which depends on the concentration, and the "transient" or "dynamic" component, which depends on the rate of change of concentration, and is zero when the rate of change of concentration is zero. That is,

$$\tau = \tau_s(f) + \tau_d[(\partial f/\partial t)_s, f] \tag{3}$$

The static stress, τ_s, is zero below the critical concentration, but the dynamic stress exists at all concentrations when the concentration is changing. Obviously, the dynamic compression resistance will be very small when the particles are far apart, but, small or not, it seems certain that it exists.

Thus the definitions of "free settling" and "compression" need to be reconsidered. The critical concentration is the concentration below which no permanent or static stress acts. Free settling is settling at concentrations below the critical, but it no longer implies absence of compressive stress. In free settling transient compressive stresses can act. In compression, both permanent* and transient compressive stresses act.

*In referring to the "static" or "permanent" component of the compressive stress, it is not implied, of course, that, on removing the applied stress, the sludge returns to the critical concentration. Especially for flocculated materials, the compression process is almost completely inelastic. The existence of a static stress in the compression zone means that, to reach a given concentration, a certain stress must be applied for sufficient time. Irrespective of whether this stress is subsequently removed, compression to a higher concentration requires application of a corresponding higher stress for sufficient time.

The existence of transient compression resistance has been recognized by workers in *Soil Mechanics* (*14*). One of the simplest cases to analyze is one-dimensional consolidation by compressing a layer of soil between two porous plates. As the soil compresses, liquid expelled from the pores flows out through the porous plates.

If this process is analyzed (*14*), ignoring dynamic compression resistance, one nevertheless finds that there is a time delay in the compression process. This is caused by the drag of the liquid as it flows out of the soil layer. Because of this drag, the applied load does not act immediately throughout the soil structure, because it is partly balanced by the liquid drag. Only when the consolidation is complete, and liquid flow has ceased, does the applied stress act throughout the soil structure. Thus, in this analysis, the time delay is due to a delay in application of the applied compressive stress to each plane in the solids structure rather than to a delay in the response of the structure to the applied stress.

However, in experimental tests on soils in which measurements are made of the liquid pressure in the pores, it is found in many cases that, even after the liquid pressure has dropped virtually to zero throughout, indicating that the applied stress acts throughout the particle structure, considerable further consolidation occurs ("secondary consolidation"). Taylor (*14*) pointed out that this indicates a transient effect in the compression process ("plastic time lag" and "plastic structural resistance to compression").

A similar situation exists in analyzing the more complex process of gravity consolidation of a sludge (*8*). Even neglecting dynamic compression effects, the compression process still exhibits a time delay due to the transient support of part of the weight of the particles by the liquid displaced upward during the compression process.

THEORETICAL SIGNIFICANCE OF TRANSIENT COMPRESSION RESISTANCE

Once the existence of compression resistance in "free" settling is accepted, the anomalies in the theory, demonstrated above, disappear. The existence of compression resistance means that particles cannot pass through a concentration discontinuity because this would mean doing work at an infinite rate in compressing the suspension. (In the original model of free settling, no work is required for compression, so that sudden jumps in concentration are possible.) Thus, in the first example, there will be no discontinuity between free settling and compression zones, the whole of the suspension being in compression. The boundary conditions are zero velocity on the container bottom and fixed concentration at the top of the suspension (since the compressive stress is zero there, due to the absence of particles above to exert a stress).

In the second (fluidization) example, inclusion of dynamic stress in the equations removes the instability, as shown in the Appendix. [Although the complete answer has perhaps still not been obtained, it seems that the occurrence of aggregative fluidization is, in fact, the result of instability in the flow. However, the instabilities arise from two- or three-dimensional effects (*15*, *16*). The present discussion has been restricted to one-dimensional situations which can be achieved by using a high resistance support plate and a small diameter vessel.]

Having argued on physical grounds that all slurries will exhibit a transient compression resistance at all particle concentrations, and having shown that inclusion of transient compression resistance removes anomalies in the theory of free settling, the question which now arises is whether it is always necessary to take the effect into account.

In the compression concentration range, omitting dynamic stress does not cause theoretical difficulties, due to absence of boundary conditions or instability of the equation solutions. However, as shown by data for soils, some materials do exhibit significant transient compression effects, and so these should be taken into account for accurate characterization. Nevertheless, methods for accounting for the effect do not seem to be well established in *Soil Mechanics*. The reason for this presumably is that the inaccuracy in typical experimental data does not warrant the use of more complex theory. Inaccuracies arise due to the effect of variable particle packing, as discussed above, which is a basic difficulty in fluid/particle studies. The same situation applies in sludge thickening, and it remains to be seen whether it is necessary and practicable to take transient compression effects into account in data analysis and equipment design.

In the free-settling range, transient compression effects must be taken into account if the changes occurring in a nonuniform suspension are being studied, because the equations cannot be solved otherwise. However, the settling of an initially uniform suspension, starting in free settling, is a case where the effect can be neglected. The action of the compression resistance is to prevent the formation of a discontinuity at the top of the sediment. However, since the transient resistance will certainly be very small in the free-settling range, a concentration gradient will only extend for a small distance into the initial-concentration zone. Hence, in this case, treating the initial suspension as free settling in the old sense, with a concentration and velocity discontinuity at the interface with the sediment, will be a satisfactory approximation.

A similar approximation will be applicable in analyzing steady-state, continuous thickening. The depth required for thickening to f_c will be very small and negligible, so that the analysis given previously (*2*) can be used. There is one difference, however. Whereas it was concluded previously that the operating line must lie below the flux line in the range from f_c to f_u, but not necessarily in the range f_f to f_c, the existence of compression resistance at all concentrations indicates that the operating line must lie below the flux line in the whole range f_f to f_u.

CONCLUSIONS

Recognition that slurries exhibit an at least transient resistance to compression at all concentrations removes anomalies in the theory of free settling. Nevertheless, the effect is expected to be very small in free settling, and negligible in situations where the free-settling zone is of uniform concentration, so that the concentration gradient will be limited to a narrow zone above the sediment. In compression, transient compression resistance is usually neglected, compared to the static resistance, but it remains to be determined if this is always a satisfactory approximation.

APPENDIX. FLUIDIZATION STABILITY ANALYSIS

Dynamic Stress Omitted

Using a Lagrangian spatial coordinate, s, based on the particle motion, and dilution, r, rather than particle volume fraction, the one-dimensional material- and momentum-balance equations are (*1*)

$$\left(\frac{\partial r}{\partial t}\right)_s = \left(\frac{\partial v}{\partial s}\right)_t \tag{4}$$

and Eq. (2), and the boundary condition is $v = 0$ at $s = S$.

In the situation being considered, the liquid upflow is held constant, so that volumetric flux, ϕ_t, is constant with respect to both t and s. Thus v can be replaced by u in Eqs. (4) and (2). At steady state at the given upflow rate, u and r are uniform and equal to $\bar{u}$ and $\bar{r}$, respectively. Linearizing Eqs. (4) and (2) for small deviations u' and r' of u and r from the steady-state values gives

$$\left(\frac{\partial r'}{\partial t}\right)_s = \left(\frac{\partial u'}{\partial s}\right)_t \tag{5}$$

and

$$\left(\frac{\partial u'}{\partial t}\right)_s = -au' + br' \tag{6}$$

where a and b are positive constants. Taking Laplace transforms with respect to t (*17*):

$$pR - r'(0, s) = dU/ds \tag{7}$$

and

$$pU = -aU + bR \tag{8}$$

taking the initial condition to be particles stationary ($u = \bar{u}$) but dilution displaced from the steady-state value. Eliminating R between Eqs. (7) and (8):

$$\frac{-b}{p(p + a)}\frac{dU}{ds} + U = \frac{br'(0, s)}{p(p + a)} \tag{9}$$

The solution of this ordinary differential equation depends on the initial concentration-deviation distribution, $r'(0, s)$. The simplest case is r' = constant = A, say. The solution of Eq. (9) is then:

$$U = \frac{bA}{p(p + a)}\{1 - \exp[-p(p + a)(S - s)/b]\} \tag{10}$$

satisfying the condition $u' = 0$ at $s = S$.

The inverse Laplace transform of Eq. (10) gives the variation of u' as a function of s and t. While this inversion cannot be carried out by the usual methods, the stability of the solution can be determined from examination of the transform. The presence of the factor $\exp[-p^2(S - s)/b]$ in the transform of U indicates poles in the right half of the complex plane (*17*) and, hence, instability. {$\text{Exp}[-p^2(S - s)/b]$ has two infinities of poles on lines parallel to the real axis at infinity.} Since the equations are linear, this result is unaffected by the choice made of the initial concentration-deviation function, $r'(0, s)$.

Dynamic Stress Included

When dynamic compressive stress, τ_d, is included in the analysis, Eq. (2) is replaced by

$$\rho\left(\frac{\partial u}{\partial t}\right)_s = F_g + F_d - \left(\frac{\partial \tau_d}{\partial s}\right)_t \tag{11}$$

τ_d is a function of r and $(\partial r/\partial t)_s$ [$= (\partial u/\partial s)_t$]. Since it is zero when $(\partial r/\partial t)_s$ is zero, and $(\partial r/\partial t)_s$ is zero at steady state, the linearized expression for τ_d takes the form

$$\tau_d = -\rho c\left(\frac{\partial r'}{\partial t}\right)_s = -\rho c\left(\frac{\partial u'}{\partial s}\right)_t \tag{12}$$

where c is a positive constant. [Although it is not relevant to the present argument, Eq. (12) is, in fact, the expected general form of the equation for τ_d, except that c is a function of f rather than a constant. This is the form which has been suggested in *Soil Mechanics* (*18*).]

Thus the linearized equations are Eq. (5) and

$$\left(\frac{\partial u'}{\partial t}\right)_s = -au' + br' + c\left(\frac{\partial^2 u'}{\partial s^2}\right)_t \tag{13}$$

The additional boundary condition required is $(\partial u'/\partial s)_t = 0$ at $s = 0$ (constant concentration at the top of the suspension).

Proceeding as above, the equation obtained after taking Laplace transforms and eliminating R is

$$\frac{-c}{p+a}\frac{d^2U}{ds^2} - \frac{b}{p(p+a)}\frac{dU}{ds} + U = \frac{br'(0, s)}{p(p+a)} \tag{14}$$

For $r'(0, s) = A =$ constant, the solution for U, satisfying the two boundary conditions, is

$$U = \frac{bA}{p(p+a)}\left[1 - \frac{m_1 e^{m_2 s} - m_2 e^{m_1 s}}{m_1 e^{m_2 S} - m_2 e^{m_1 S}}\right] \tag{15}$$

where

$$m_1, m_2 = -\frac{0.5b}{cp}\left(1 \pm \sqrt{1 + \frac{4cp^2(p+a)}{b^2}}\right)$$

It is not difficult to show that this expression for U contains no poles in the right half of the complex plane, and so the solution is stable.

SYMBOLS

The positive direction is downward for all vector quantities.

- a linearization constant $= -(\partial F_d/\partial u)_r/\rho$, sec^{-1}
- A amplitude of initial dilution disturbance, dimensionless
- b linearization constant $= (\partial F_d/\partial r)_u/\rho$, m/sec^2

c	linearization constant $= -(\partial\tau_d/\partial(\partial r/\partial t)_s)_r/\rho$, m^2/sec
f	particle concentration, volume fraction, dimensionless
f_c	critical particle concentration
f_f	feed particle concentration
f_u	underflow particle concentration
F_d	liquid drag force on the particles, per unit volume of particles, N/m^3
F_g	$g(\rho - \rho_l)$ = net gravitational force, per unit volume of particles, N/m^3
g	acceleration due to gravity, m/sec^2
m_1, m_2	expression involved in solution of Eq. (14), m^{-1}
p	Laplace transform variable, sec^{-1}
r	particle dilution $= 1/f$, dimensionless
$\bar{r}$	steady-state value of r
r'	deviation of r from $\bar{r}$
R	Laplace transform of r', sec
s	volume of particles per unit cross-sectional area, below a reference plane moving with the particles, m
S	total volume of particles in the system per unit cross-sectional area, m
t	time, sec
u	velocity of particles, relative to slurry volume-average velocity $= v - \phi_t$, m/sec
$\bar{u}$	steady-state value of u
u'	deviation of u from $\bar{u}$
U	Laplace transform of u', m
v	velocity of particles, relative to container, m/sec

Greek

ρ	particle density, kg/m^3
ρ_l	liquid density, kg/m^3
τ	interparticle compressive stress, based on total cross-sectional area, N/m^2
τ_s	static component of τ
τ_d	dynamic component of τ
ϕ_t	total volumetric flux = volume-average velocity, m/sec

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Received by editor March 1, 1978